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## Ground state measure and its applications

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### 1 Introduction

In this paper we shall consider structures of ground states of a model describing an interaction between a particle and a quantized scalar bose field, which is called the “Nelson model” [15],[18]. Basic ideas in this paper is due to a fairly nice work of H.Spohn [22], in which he studies the spin-boson model. The Hamiltonian,  $H$ , of the Nelson model is defined as a self-adjoint operator acting on Hilbert space  $\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F}$ , where  $\mathcal{F}$  denotes a Boson Fock space. The existence of the ground states,  $\Psi_g$ , of  $H$  is established in e.g., [2],[4],[12],[23]. The main results presented here is to give the expectation-value of the number of bosons of  $\Psi_g$  and its boson distribution by means of a ground state measure constructed in this paper. Especially the localization of bosons of  $\Psi_g$  is proved. The ground state measure,  $\mu$ , on the set of paths,  $\Omega$ , gives an integral representation of the expectation-value of certain operator  $A$  in  $\mathcal{H}$ , i.e.,

$$(\Psi_g, A\Psi_g) = \int_{\Omega} f_A(q)\mu(dq),$$

where  $f_A$  is a density function corresponding to  $A$ . This integral representation leads us to the goal of this paper. Detailed arguments shall be published elsewhere [2], and refer to see [17],[21],[22]. This paper is organized as follows: section 2 gives a definition of models considered in this paper. In section 3 we review the second quantizations. Section 4 is devoted to investigating the ground states. In section 5 we give further problems on the Pauli-Fierz model in nonrelativistic quantum electrodynamics.

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## 2 Scalar quantum field models

Let  $\mathcal{F} := \bigoplus_{n=0}^{\infty} \bigotimes_s^n L^2(\mathbb{R}^d) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n$ , where  $\bigotimes_s^n$  denotes the  $n$ -fold symmetric tensor product with  $\bigotimes_s^0 L^2(\mathbb{R}^d) := \mathbb{C}$ . The bare vacuum,  $\Omega \in \mathcal{F}$ , is defined by  $\Omega := \{1, 0, 0, \dots\}$ . Let  $a^\dagger(f)$  and  $a(g)$  be the creation operator and the annihilation operator smeared by  $f, g \in L^2(\mathbb{R}^d)$ , respectively, which are linear in  $f$  and  $g$ . Let  $\mathcal{F}_{\text{fin}}$  be the finite particle subspace of  $\mathcal{F}$ :

$$\mathcal{F}_{\text{fin}} := \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F} \mid \text{there exists } n_0 \text{ such that } \Psi^{(m)} = 0, m \geq n_0 \right\}.$$

They satisfy canonical commutation relations (CCR), i.e.,

$$[a(f), a^\dagger(g)] = (\bar{f}, g)_{L^2(\mathbb{R}^d)}, \quad [a^\sharp(f), a^\sharp(g)] = 0,$$

on  $\mathcal{F}_{\text{fin}}$ , where  $a^\sharp$  denotes  $a$  or  $a^\dagger$ , and  $(\cdot, \cdot)_{\mathcal{K}}$  the scalar product on Hilbert space  $\mathcal{K}$ . We denote by  $\|\cdot\|_{\mathcal{K}}$  its associated norm. Unless confusion arises we omit  $\mathcal{K}$  in  $(\cdot, \cdot)_{\mathcal{K}}$  and  $\|\cdot\|_{\mathcal{K}}$ , respectively.  $a^\sharp$  also satisfies that  $(\Psi, a(f)\Phi) = (a^\dagger(\bar{f})\Psi, \Phi)$  for  $\Psi, \Phi \in \mathcal{F}_{\text{fin}}$ . For dense subset  $\mathcal{K} \subset L^2(\mathbb{R}^d)$ ,

$$\mathcal{F}(\mathcal{K}) := l.h.\{a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega, \Omega \mid f_j \in \mathcal{K}, j = 1, \dots, n, n \in \mathbb{N}\}$$

is dense in  $\mathcal{F}$ . We define the free Hamiltonian,  $H_f$ , in  $\mathcal{F}$  by

$$H_f \Omega := 0,$$

$$H_f a^\dagger(f_1) \cdots a^\dagger(f_n) \Omega := \sum_{j=1}^n a^\dagger(f_1) \cdots a^\dagger(\omega f_j) \cdots a^\dagger(f_n) \Omega,$$

$$f_j \in D(\omega), \quad j = 1, \dots, n, \quad n \in \mathbb{N},$$

where  $D(T)$  denotes the domain of  $T$ ,  $\omega := \omega(k) := \sqrt{|k|^2 + m^2}$ ,  $m \geq 0$ . Here  $m$  denotes the mass of the quantized scalar bose field. Field operators  $\phi(f)$  are defined by

$$\phi(f) := \frac{1}{\sqrt{2}}(a^\dagger(\bar{f}) + a(f)), \quad f \in L^2(\mathbb{R}^d).$$

Note that  $H_f|_{\mathcal{F}(D(\omega))}$  and  $\phi(f)|_{\mathcal{F}_{\text{fin}}}$  are essentially self-adjoint, respectively. It is known that  $\sigma(H_f) = [0, \infty)$  and  $\sigma_p(H_f) = \{0\}$ . The Hamiltonian,  $H$ , considered in this paper is defined by

$$H := H_p \otimes 1 + 1 \otimes H_f + \alpha H_I$$

on  $\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F} \cong L^2(\mathbb{R}^d; \mathcal{F})$ , where  $\alpha \in \mathbb{R}$  is a coupling constant, and

$$H_{\text{I}} := \phi(e^{ikx} \hat{\lambda}),$$

$$H_{\text{p}} := -\Delta/2 + V,$$

where  $\hat{\lambda}$  is the Fourier transform of  $\lambda$ . A reasonable physical choice of  $\hat{\lambda}$  is of the form

$$\hat{\lambda} = \hat{\rho} / \sqrt{(2\pi)^d \omega},$$

where  $\rho$  describes a charge distribution, i.e.,

$$\sqrt{(2\pi)^d} \hat{\rho}(0) = \int_{\mathbb{R}^d} \rho(x) dx = \alpha.$$

For simplicity we assume that external potential  $V = V_+ - V_-$  satisfies that  $V_+ \in L^1_{\text{loc}}(\mathbb{R}^d)$  and that  $V_-$  is infinitesimally small with respect to  $\Delta$  in the sense of form. Throughout this paper we assume that

$$\overline{\hat{\lambda}(k)} = \hat{\lambda}(-k).$$

Let  $\hat{\lambda}, \hat{\lambda}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$ . Then it is known that, for arbitrary  $\alpha$ ,  $H$  is self-adjoint on  $D(H_{\text{p}} \otimes 1) \cap D(1 \otimes H_{\text{f}})$  and bounded from below. Moreover it is essentially self-adjoint on any core of  $H_{\text{p}} \otimes 1 + 1 \otimes H_{\text{f}}$ .

**Proposition 2.1** ([2],[12]) *Let  $\hat{\lambda}/\omega, \hat{\lambda}/\sqrt{\omega}, \hat{\lambda} \in L^2(\mathbb{R}^d)$ . Then there exists  $\alpha_*$  such that for  $|\alpha| \leq \alpha_*$  the ground states,  $\Psi_g$ , of  $H$  exist. Moreover  $(f \otimes \Omega, \Psi_g) > 0$  for arbitrary nonnegative  $f \in L^2(\mathbb{R}^d)$  with  $f \not\equiv 0$ .*

See Figure 2 for more explicit results on the existence of the ground states of  $H$ .

### 3 Second quantizations

For later use we review the second quantization of operator  $T$  on  $L^2(\mathbb{R}^d)$ . Let  $T$  be a contraction operator on  $L^2(\mathbb{R}^d)$ , i.e.,  $\|T\| \leq 1$ . Then we define  $\Gamma(T) : \mathcal{F}_{\text{fin}} \rightarrow \mathcal{F}_{\text{fin}}$  by

$$\Gamma(T)\Omega := \Omega,$$

$$\Gamma(T)a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega := a^\dagger(Tf_1)\cdots a^\dagger(Tf_n)\Omega,$$

$$f_j \in L^2(\mathbb{R}^d), \quad j = 1, \dots, n, \quad n \in \mathbb{N}.$$

For  $\Phi \in \mathcal{F}_{\text{fin}}$  we have  $\|\Gamma(T)\Phi\| \leq \|\Phi\|$ . Thus  $\Gamma(T)$  extends to a contraction operator on  $\mathcal{F}$ . We denote its extension by the same symbol. It is seen that  $\Gamma(\cdot)$  is linear in  $\cdot$  and that  $\Gamma(T)^* = \Gamma(T^*)$ . Let  $h$  be a nonnegative self-adjoint operator in  $L^2(\mathbb{R}^d)$ . Then we see that  $\Gamma(e^{-th})$  is a strongly continuous symmetric contraction one-parameter semigroup in  $t \geq 0$ . The second quantization of  $h$ ,  $d\Gamma(h)$ , is defined by the generator of  $\Gamma(e^{-th})$ , i.e.,

$$\Gamma(e^{-th}) = e^{-td\Gamma(h)}, \quad t \geq 0.$$

Actually  $H_f$  is the second quantization of multiplication operator  $\omega$ . For nonnegative multiplication operator  $h$  in  $L^2(\mathbb{R}^d)$ , formally, it is written as

$$d\Gamma(h) = \int h(k)a^\dagger(k)a(k)dk. \quad (3.1)$$

The number operator,  $N$ , in  $\mathcal{F}$  is defined by the second quantization of the identity operator in  $L^2(\mathbb{R}^d)$ , i.e.,

$$D(N) := \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F} \left| \sum_{n=0}^\infty n^2 \|\Psi^{(n)}\|_{\mathcal{F}_n}^2 < \infty \right. \right\},$$

$$(N\Psi)^{(n)} := n\Psi^{(n)}.$$

Let  $h$  be a multiplication operator in  $L^2(\mathbb{R}^d)$  such that  $s = s_{R+} - s_{R-} + i(s_{I+} - s_{I-})$ , where  $s_{R+}$  (resp.  $s_{R-}, s_{I+}, s_{I-}$ ) denotes the real positive (resp. real nonpositive, imaginary positive, imaginary nonpositive) part of  $s$ . Then we define

$$d\Gamma(h) := d\Gamma(s_{R+}) - d\Gamma(s_{R-}) + i(d\Gamma(h_{I+}) - d\Gamma(h_{I-})),$$

$$D(d\Gamma(h)) := D(d\Gamma(s_{R+})) \cap D(d\Gamma(s_{R-})) \cap D(d\Gamma(h_{I+})) \cap D(d\Gamma(h_{I-})).$$

## 4 Ground state measures

Let  $\Omega := (\mathbb{R}^d)^{(-\infty, \infty)}$  be the set of  $\mathbb{R}^d$ -valued paths and  $\mathcal{B}(\Omega)$  the  $\sigma$ -field constructed by cylinder sets. For  $T : \mathcal{H} \rightarrow \mathcal{H}$ , we define

$$\langle T \rangle := (\Psi_g, T\Psi_g)_{\mathcal{H}}.$$

For a convenience we denote by  $\langle S \rangle$  for  $\langle 1 \otimes S \rangle$ , for  $S : \mathcal{F} \rightarrow \mathcal{F}$ . Our fundamental theorem is as follows:

**Theorem 4.1 ([2])** *Let  $s$  be such that  $\sup_{k \in \mathbb{R}^d} |s(k)| < \infty$ . Let  $\hat{\lambda}/\omega$ ,  $\hat{\lambda}/\sqrt{\omega}$ ,  $\hat{\lambda} \in L^2(\mathbb{R}^d)$ , and  $|\alpha| \leq \alpha_*$ . We assume that  $A_1, \dots, A_m$  are measurable sets in  $\mathbb{R}^d$  and let  $1_A$  denote the characteristic function of  $A$ . Then there exists a probability measure  $\mu$  on  $(\Omega, \mathcal{B}(\Omega))$  such that, for  $t_1 \leq \dots \leq t_m$ ,*

$$\begin{aligned} \langle 1_{A_1} e^{-(t_2-t_1)H} 1_{A_2} \dots e^{-(t_m-t_{m-1})H} 1_{A_m} \rangle &= \int_{\Omega} 1_{A_1}(q(t_1)) \dots 1_{A_m}(q(t_m)) \mu(dq), \\ \langle e^{-\beta d\Gamma(s)} \rangle &= \int_{\Omega} e^{(\alpha^2/2)Z(\beta)} \mu(dq), \quad \beta > 0, \end{aligned} \quad (4.1)$$

where

$$Z(\beta) := \int_{-\infty}^0 dt \int_0^{\infty} ds \int_{\mathbb{R}^d} |\hat{\lambda}(k)|^2 e^{-|t-s|\omega(k)} (e^{-\beta s(k)} - 1) e^{ik(q(t)-q(s))} dk.$$

We give a remark on  $Z(\beta)$ . Since  $\|\hat{\lambda}/\omega\| < \infty$ , we see that

$$|Z(\beta)| \leq 2 \|\hat{\lambda}/\omega\|^2 < \infty$$

uniformly in paths  $q \in \Omega$ . Thus  $Z(\beta)$  is well defined. It is proved in [2] that  $\mu$  is a Gibbs measure. We call  $\mu$  the “ground state measure for  $H$ ”. It is easily seen that the right-hand side of (4.1) is analytically continued to  $\beta \in \mathbb{C}$ . Although it does *not* imply that  $\langle e^{-\beta d\Gamma(s)} \rangle$  is well defined for all  $\beta \in \mathbb{C}$ , we have the following theorem:

**Theorem 4.2 ([2])** *Let  $s$ ,  $\hat{\lambda}$  and  $\alpha$  be in Theorem 4.1. Then we have  $\Psi_g \in D(1 \otimes e^{-\beta d\Gamma(s)})$  for all  $\beta \in \mathbb{C}$ , and (4.1) holds true for all  $\beta \in \mathbb{C}$ .*

We immediately have the following corollary.

**Corollary 4.3** *Let  $\hat{\lambda}$  and  $\alpha$  be in Theorem 4.1. Then, for arbitrary  $\epsilon \in \mathbb{R}$ , we have  $\Psi_g \in D(1 \otimes e^{\epsilon N})$ . Moreover*

$$\langle N \rangle = \frac{\alpha^2}{2} \int_{-\infty}^0 dt \int_0^{\infty} ds \int_{\mathbb{R}^d} dk |\hat{\lambda}(k)|^2 e^{-|t-s|\omega(k)} \int_{\Omega} e^{ik(q(t)-q(s))} \mu(dq). \quad (4.2)$$

*Proof:* Putting  $s = 1$  in Theorem 4.2, we get  $\Psi_g \in D(1 \otimes e^{\epsilon N})$  for all  $\epsilon \in \mathbb{R}$ . (4.2) follows from (4.1) and

$$\langle N \rangle = - \frac{d \langle e^{-\beta N} \rangle}{d\beta} \Big|_{\beta=0}.$$

The proof is complete. Q.E.D.

Corollary 4.3 implies that

$$\sum_{n=0}^{\infty} e^{2\epsilon n} \|\Psi_g^{(n)}\|_{L^2(\mathbb{R}^d) \otimes \mathcal{F}_n}^2 < \infty, \quad \text{for all } \epsilon > 0.$$

Hence we conclude that  $\|\Psi_g^{(n)}\|$  decays super-exponentially as  $n \rightarrow \infty$ ; it decays faster than  $e^{-\epsilon n}$  for arbitrary  $\epsilon > 0$ . Let  $s \in C_0^\infty(\mathbb{R}^d)$ . Then, by Theorem 4.2, we see that  $\Psi_g \in D(d\Gamma(s))$  and

$$|\langle d\Gamma(s) \rangle| \leq (\alpha^2/2) \|s\|_\infty \|\hat{\lambda}/\omega\|^2.$$

Thus map

$$\mathcal{D} : C_0^\infty(\mathbb{R}^d) \ni s \rightarrow \langle d\Gamma(s) \rangle \in \mathbb{C}$$

defines a distribution on  $C_0^\infty(\mathbb{R}^d)$ . Taking into account of the formal expression of  $d\Gamma(s)$  (3.1), we denote by  $\langle a^\dagger(k)a(k) \rangle$  the kernel of  $\mathcal{D}$ . From Corollary 4.3 it immediately follows:

**Corollary 4.4** *Let  $\hat{\lambda}$  and  $\alpha$  be in Theorem 4.1. Then for a.e.  $k \in \mathbb{R}^d$ ,*

$$\langle a^\dagger(k)a(k) \rangle = \frac{\alpha^2}{2} |\hat{\lambda}(k)|^2 \int_{-\infty}^0 dt \int_0^\infty ds e^{-|t-s|\omega(k)} \int_{\Omega} e^{ik(q(t)-q(s))} \mu(dq).$$

Note that

$$\int_{\mathbb{R}^d} \langle a^\dagger(k)a(k) \rangle dk = \langle N \rangle.$$

Moreover we see that

$$|\langle a^\dagger(k)a(k) \rangle| \leq \frac{\alpha^2}{2} \frac{|\hat{\lambda}(k)|^2}{\omega(k)^2}, \quad \text{a.e. } k \in \mathbb{R}^d.$$

See Figure 1.

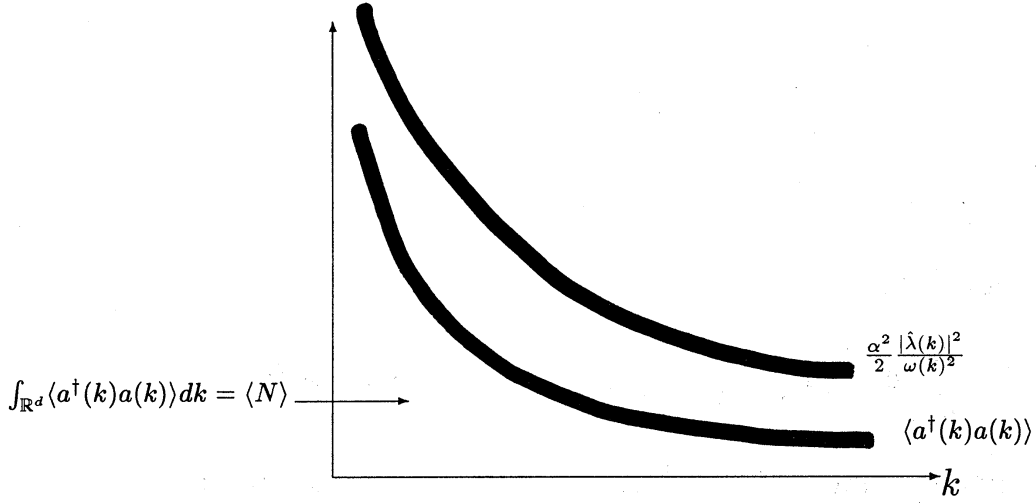


Figure 1: Infrared cutoff  $\|\hat{\lambda}/\omega\| < \infty$  and  $\langle N \rangle < \infty$

## 5 Nonrelativistic QED

### 5.1 The Pauli-Fierz model

The Pauli-Fierz model [1],[3],[5]-[11],[19],[20] in nonrelativistic QED describes an interaction of particles (electrons) and a quantized radiation field (photons). The quantized radiation field is quantized in a Coulomb gage. We assume that the number of the electrons is one and that the electron has spineless. Let

$$\mathcal{F}_{\text{PF}} := \bigoplus_{n=0}^{\infty} \bigotimes_s^n \underbrace{L^2(\mathbb{R}^d) \oplus \cdots \oplus L^2(\mathbb{R}^d)}_{d-1} \cong \underbrace{\mathcal{F} \otimes \cdots \otimes \mathcal{F}}_{d-1}.$$

Let  $\{b^r(f), b^{\dagger r}(g)\}_{r=1}^{d-1}$  be the annihilation operators and the creation operators, respectively, which satisfy CCR:

$$[b^r(f), b^{\dagger s}(g)] = \delta_{rs}(\bar{f}, g)_{L^2(\mathbb{R}^d)}, \quad [b^{\sharp r}(f), b^{\sharp s}(g)] = 0.$$

Let  $H_{\text{f}}^{\text{PF}}$  be the free Hamiltonian in  $\mathcal{F}_{\text{PF}}$ , i.e.,

$$H_{\text{f}}^{\text{PF}} := \sum_{r=1}^{d-1} \int \omega(k) b^{\dagger r}(k) b^r(k) dk.$$



The Hamiltonian of the Pauli-Fierz model is defined as an operator in  $\mathcal{H}_{\text{PF}} := L^2(\mathbb{R}^d) \otimes \mathcal{F}_{\text{PF}} \cong L^2(\mathbb{R}^d; \mathcal{F}_{\text{PF}})$  and reads

$$H_{\text{PF}} := \frac{1}{2} (-i\nabla \otimes 1 - e\mathbf{A}(x))^2 + 1 \otimes H_{\text{f}}^{\text{PF}} + V \otimes 1,$$

where  $e$  is a coupling constant,  $\mathbf{A}(x) := (\mathbf{A}_1(x), \dots, \mathbf{A}_d(x))$ ,

$$\mathbf{A}_\mu(x) := \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} \left( b^{\dagger r} (e_\mu^r \bar{\lambda} e^{-ikx}) + b^r (e_\mu^r \lambda e^{ikx}) \right),$$

and  $e^r := (e_1^r, \dots, e_d^r)$ , polarization vectors;  $e^r(k) \cdot e^s(k) = \delta_{rs}$  and  $e^r(k) \cdot k = 0$ . Note that

$$\text{div} \mathbf{A} = 0.$$

For the Nelson model, the self-adjointness of  $H$  for arbitrary  $\alpha$  is trivial, since  $H_{\text{I}}$  is infinitesimally small with respect to  $H_{\text{p}} \otimes 1 + 1 \otimes H_{\text{f}}$ . It is not so easy to show self-adjointness of  $H_{\text{PF}}$  for *arbitrary*  $e \in \mathbb{R}$ . Let  $N_{\text{PF}}$  be the number operator in  $\mathcal{F}_{\text{PF}}$ . We have the following proposition:

**Proposition 5.1** ([9]) <sup>1</sup> *Let  $\hat{\lambda}, \omega^2 \hat{\lambda} \in L^2(\mathbb{R}^d)$ . We assume that  $V$  is relatively bounded with respect to  $\Delta$ . Then, for arbitrary  $e \in \mathbb{R}$ ,  $H_{\text{PF}}$  is essentially self-adjoint on*

$$D(\Delta \otimes 1) \cap D(1 \otimes (H_{\text{f}}^{\text{PF}})^2) \cap_{k=1}^{\infty} D(1 \otimes N_{\text{PF}}^k).$$

The existence of the ground states of  $H_{\text{PF}}$  are studied in [1],[6], and their multiplicities in [7],[11]. Moreover  $\inf \sigma(H_{\text{PF}})$  is investigated in [3],[16].

## 5.2 Ground states of $H$ and $H_{\text{PF}}$

Let

$$\text{gap}(T) := \inf \sigma_{\text{ess}}(T) - \inf \sigma(T).$$

The existence of the ground states of  $H$  and  $H_{\text{PF}}$  are deeply related to conditions on  $m$ ,  $\text{gap}$ ,  $\hat{\lambda}$  and coupling constants. Let  $\hat{\lambda}/\omega \in L^2(\mathbb{R}^d)$ .<sup>2</sup> Then sufficient conditions for the existence of the ground states of  $H$  and  $H_{\text{PF}}$ , as far as we know, are in Figures 2 and 3, respectively.

	$m > 0$	$m = 0$
$\text{gap}(H) = \infty$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{R}$
$0 < \text{gap}(H) < \infty$	$ \alpha  \ll 1$	$ \alpha  \ll 1$

Figure 2:  $\alpha$  for the existence of the ground states of  $H$ .

	$m > 0$	$m = 0$
$\text{gap}(H_{\text{PF}}) = \infty$	$e \in \mathbb{R}$	$ e  \ll 1$
$0 < \text{gap}(H_{\text{PF}}) < \infty$	$ e  \ll 1$	$ e  \ll 1$

Figure 3:  $e$  for the existence of the ground states of  $H_{\text{PF}}$ .

Note that see [4],[23] for a proof of the existence of ground states for case  $\text{gap}(H) = \infty$  and  $m \geq 0$  in Figure 2, and [8],[9] for case  $\text{gap}(H_{\text{PF}}) = \infty$  and  $m > 0$  in Figure 3. In [13],[14] the authors give examples such that the ground states of  $H$  and  $H_{\text{PF}}$  exist for the case where  $\text{gap}(H) = 0$  and  $\text{gap}(H_{\text{PF}}) = 0$ , respectively. In [17] no existence of the ground states of  $H$  for arbitrary  $\alpha \neq 0$  is proved if  $\|\hat{\lambda}/\omega\| = \infty$ .

### 5.3 Distribution of bosons for $\Psi_{\text{PF}}$

Let  $\Psi_{\text{PF}}$  be the ground state of  $H_{\text{PF}}$  and

$$\langle T \rangle_{\text{PF}} := (\Psi_{\text{PF}}, T \Psi_{\text{PF}})_{\mathcal{H}_{\text{PF}}}.$$

<sup>1</sup>In [9] essential self-adjointness of  $H_{\text{PF}}$  is proved only for the case where the number of the electrons is *one*. As far as we know it is not clear whether the statement in Proposition 5.1 with  $N$ -electrons holds true or not. In [19] self-adjointness of  $H_{\text{PF}}$  on  $D(\Delta \otimes 1) \cap D(1 \otimes H_{\text{f}}^{\text{PF}})$  is proved for sufficiently small  $|e|$ .

<sup>2</sup>It is not necessarily to assume  $\hat{\lambda}/\omega \in L^2(\mathbb{R}^d)$  for  $H_{\text{PF}}$ . See [1].

Our next problem is to study the distribution of bosons of  $\Psi_{\text{PF}}$ , e.g.,  $\langle N_{\text{PF}} \rangle_{\text{PF}}$ ,  $\langle e^{-\beta N_{\text{PF}}} \rangle_{\text{PF}}$ , etc. In [10] a ground state measure,  $\mu_{\text{PF}}$ , on  $(\Omega, \mathcal{B}(\Omega))$  for  $H_{\text{PF}}$  is constructed, which satisfies

$$\begin{aligned} & \langle 1_{A_1} e^{-(t_2-t_1)H_{\text{PF}}} 1_{A_2} \cdots e^{-(t_m-t_{m-1})H_{\text{PF}}} 1_{A_m} \rangle_{\text{PF}} \\ &= \int_{\Omega} 1_{A_1}(q(t_1)) \cdots 1_{A_m}(q(t_m)) \mu_{\text{PF}}(dq). \end{aligned}$$

Moreover a “formal” calculation gives a “formal” expression [5],[21]:

$$\langle e^{-\beta N_{\text{PF}}} \rangle_{\text{PF}} = \int_{\Omega} e^{(-e^2/2)Z_{\text{PF}}(\beta)} \mu_{\text{PF}}(dq),$$

where

$$\begin{aligned} Z_{\text{PF}}(\beta) &:= (e^{-\beta} - 1) \sum_{\mu, \nu=1}^d \int_{-\infty}^0 dq_{\mu}(t) \int_0^{\infty} dq_{\nu}(s) \times \\ &\times \int_{\mathbb{R}^d} d_{\mu\nu}(k) |\hat{\lambda}(k)|^2 e^{-|t-s|\omega(k)} e^{ik(q(t)-q(s))} dk. \end{aligned}$$

Here  $d_{\mu\nu}(k) := \sum_{r=1}^{d-1} e_{\mu}^r(k) e_{\nu}^r(k)$  and  $\int \cdots dq_{\mu}(t)$  denotes a stochastic integral. For the Nelson model  $|Z(\beta)| \leq 2 \|\hat{\lambda}/\omega\|^2 < \infty$  guarantees that  $\int_{\Omega} e^{(\alpha^2/2)Z(\beta)} \mu(dq)$  is well defined. We do not have such an estimate for  $Z_{\text{PF}}(\beta)$ , which is a crucial points to study  $\langle N_{\text{PF}} \rangle_{\text{PF}}$  in terms of the ground state measure. Actually the definition of  $Z_{\text{PF}}(\beta)$  is not clear, e.g., it is needed to give a rigorous definition of  $\int_{-\infty}^0 dq_{\mu}(t) \int_0^{\infty} dq_{\nu}(s)$ .

## 5.4 Conjectures and problems

In view of subsections 5.1-5.3, we give the following conjectures. We assume some conditions on  $\hat{\lambda}$  and  $V$ .

**Conjecture 5.2** *For arbitrary  $e \in \mathbb{R}$ ,  $H_{\text{PF}}$  is self-adjoint and bounded from below on  $D(\Delta \otimes 1) \cap D(1 \otimes H_{\text{f}}^{\text{PF}})$ .*

**Conjecture 5.3** *Let  $\text{gap}(H_{\text{PF}}) = \infty$  and  $m \geq 0$ . Then the ground states of  $H_{\text{PF}}$  exist for arbitrary  $e \in \mathbb{R}$ .*

**Conjecture 5.4**  *$\Psi_{\text{PF}} \in D(1 \otimes e^{\epsilon N_{\text{PF}}})$  for all  $\epsilon \in \mathbb{R}$ .*

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